Motion of Freely Falling Spheres at Moderate Reynolds Numbers

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A basic mechanism for the wandering of freely falling spheres is shown to be coupling between rocking of the spheres and their motions perpendicular to the free fall direction. The rocking frequency is determined by small displacements of the spheres' centers of mass from their geometric centers. A phenomenological model of the motion, in which the rocking of the spheres is described by the nonlinear pendulum equation with damping, and the coupling between the rocking and the lateral force on the spheres involves a Reynolds number dependent phase shift, leads to fairly good agreement with observations. The wandering mechanism discussed here is most effective at Reynolds numbers in the range $10^3 < Re < 10^5$, for sphere-to-fluid mass ratios between 0.8 and 1.2.

Introduction

SEVERAL observers have noted that solid spheres falling freely in a viscous fluid do not always move along straight lines. The earliest references are by Newton,[1] who discussed experiments which involved dropping hollow spheres from the dome of St. Paul's Cathedral in London. Some of the spheres were glass, while others were inflated hogs' bladders. The experiments were carefully timed, and the comparison between their results and the predictions of Newtonian fluid theory was surprisingly good. However, Newton reported that in some cases the spheres did not fall straight down, but deviated somewhat from the vertical trajectory. Newton also considered the lift force on a spinning sphere and reported some experiments with weighted wax spheres falling in water. These spheres fell with considerable swerving. According to Newton, " . . . the globes, when they were first let go and began to fall, oscillated about their centers; that side which chanced to be the heavier descending first, and producing an oscillating motion." Also, " . . . the globe always recedes from that side of itself which is descending in the oscillation, and by so receding comes nearer to the sides of the vessel, so as even to strike against them sometimes." The lift on a spinning body was also discussed by Magnus,[2] and the phenomenon is often referred to as the Magnus effect.

Two of the more novel sphere experiments were conducted by Eiffel,[3] who obtained drag coefficients by dropping spheres from the Eiffel Tower, and Richardson,[4] who fired cannon balls vertically from an unrifled cannon.

Shakespear[5] found variations in the terminal velocities of falling spheres which he attributed partly to the noncoincidence of the center of gravity and the geometric centers of the spheres. Lunnon[6] dropped spheres of various densities in the mine shafts of the Tyneside, England coal district. He found a deviation of not more than a few inches from the vertical path at the bottom of the trajectory. However, this is on the order of ten diameters of the spheres tested. Lunnon[7] also tested falling spheres in a water tank. He found that " . . . the falls were not straight. There is always some swerving in the path of falling spheres." When this swerving was too large, his experimental run was rejected.

Just prior to Lunnon's experiments, Schmidt[8] observed balloons rising in air and weighted spheres falling in water. He found that the spheres did not always stay on a vertical trajectory and in his paper considered " . . . only those spheres whose trajectories remain near the vertical through the starting point." Schmiedel[9] found similar behavior in the case of unweighted spheres. More recently Barker[10] found an apparently random wobbling of the freely falling sphere. The maximum amplitude of the motion was approximately equal to the sphere's diameter. Shafrir[11] found a similar behavior but with larger amplitudes of the lateral motion.

It is natural to suppose that the excursions of freely falling spheres reported in the literature may be due to some instability of the separated flow region behind the sphere, or to regular vortex shedding. These phenomena may very well cause wandering when the spheres fall at certain Reynolds numbers for certain ratios of the sphere and fluid densities. However, during experiments with spheres whose specific gravities were close to one, falling in water at Reynolds numbers between 3000 and 35,000, we noticed that the wandering was always associated with rocking of the spheres. Pursuing this, we have found fairly good agreement between observations and a theory in which wandering is predicted as the result of coupling between rocking of the spheres due to small displacements of their centers of mass from their geometric centers, and their lateral motion. The coupling is essentially the same phenomenon as the lateral force on a sphere spinning in a uniform flow predicted by Stokes[12] and observed by Maccoll.[12]

The wandering phenomenon described in this paper appears to be relevant to both atmospheric and oceanic sounding with spherical probes. For example, balloon wind sensors operating at Reynolds numbers in the range of our study do not rise vertically, even in a calm atmosphere.[13]

Preliminary Theory

Our experiments involved a turbulent separated flow about a bluff body in unsteady motion. In view of the complexity of this flow, we found it appropriate to develop a phenomenological theory along the following lines. If a sphere is biased, i.e., if its center of mass (c.m.) is displaced from its geometric center, in a free fall it will tend to oscillate about a preferred orientation in which its c.m. is directly below its geometric

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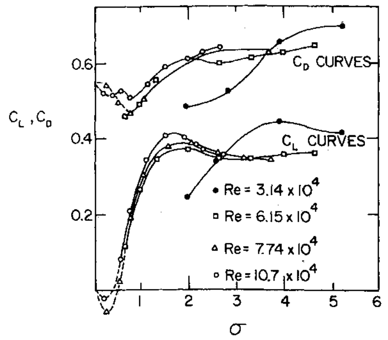


Fig. 1 Lift and drag on a steadily spinning sphere (from Maccoll).[1]

center. Quite noticeable oscillations occur even when the c.m. is displaced from the center by only a few hundredths of a radius. Also, when a sphere rotates in a uniform flow it experiences a lift force in a direction perpendicular to the uniform flow. This lifting phenomenon can couple bias-induced rocking to the lateral motion of a freely falling sphere, so that a wandering motion is observed.

Observations in the literature permit us to make this notion more precise. According to Maccoll,[12] J. J. Stokes suggested that a sphere in a uniform flow of speed U_0, rotating with angular speed ω about an axis perpendicular to the uniform flow, will experience a force of magnitude

$$F = C\omega U_o \tag{1}$$

in the direction of $\mathbf{U} \times \mathbf{\Omega}$, where $\mathbf{\Omega}$ is the sphere's angular velocity and $\mathbf{U}$ the freestream velocity. Stokes conjectured that the simple bilinear dependence of F on ω and U_o shown in Eq. (1) would be a good approximation so long as the largest tangential velocity of the sphere was smaller than U_o, i.e., so long as

$$\sigma \equiv \omega a/U_o < 1 \tag{2}$$

Maccoll[12] tested this model experimentally. His results (Fig. 1) show that, provided σ is neither too large nor too small,

$$F = \pi a^2 U_o^2 k(\sigma - \sigma_o)\rho_f/2 \tag{3}$$

Here the nondimensional constant k is the slope of Maccoll's C_L-σ curve in its linear portion. As indicated in Fig. 1, k depends on Reynolds number R_e. Provided that $\sigma \gg \sigma_o$ while still in the linear range,

$$F \approx \pi a^3 k\omega U_o\rho_f/2 = 3mk\omega U_o/(8\gamma) \tag{4}$$

where $\gamma \equiv \rho_s/\rho_f$, and m is the mass of the sphere. Thus, comparing Eqs. (4) and (1), one sees that Maccoll's observations confirm Eq. (1) for some values of σ, with

$$C = 3mk/(8\gamma) \tag{5}$$

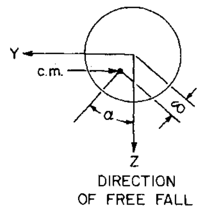
To construct a preliminary theory of the coupling between a freely falling sphere's rocking and its lateral motion, we assume that: 1) Eq. (1), subject to inequality (2), gives an acceptable approximation to the lateral force experienced by a rocking sphere, even when ω is not constant, but is a periodic function of time; 2) the rocking of a biased sphere is not itself affected by the disturbances it produces in the sphere's motion; 3) the vertical motion of the wandering sphere is negligibly different from a steady fall. This assumption was in fact satisfied in each of our experiments, after a brief initial acceleration period.

Assumption 2) permits us to treat the rocking and the lateral motion independently. For the present we will neglect damping, and assume that the rocking of a biased sphere is described by the equation

$$I\ddot{\alpha} = -\delta mg \sin\alpha \approx -\delta mg\alpha \tag{6}$$

where α is the sphere's angular displacement from its equilibrium orientation (Fig. 2), I is the moment of inertia about

Fig. 2 Description of biased sphere and its motion.



the axis of rotation, and δ is the displacement of the sphere's c.m. from its geometric center. One then finds, on solving Eq. (6), that the angular displacement of the rocking sphere is

$$\alpha(t) = \alpha_o \exp(ipt)$$

where as usual physical quantities are to be taken as the real parts of complex quantities, and where

$$p = (\delta mg/I)^{1/2}$$

Then the sphere's angular speed ω is given by

$$\omega = \dot{\alpha} = ip\alpha_o \exp(ipt) \tag{7}$$

If now y denotes the sphere's displacement in the direction of $\mathbf{U} \times \mathbf{\Omega}$, by assumption 1, its lateral motion will be described by

$$m\ddot{y} = F \tag{8}$$

or, using Eqs. (7) and (1),

$$m\ddot{y} = iCp\alpha_o U_o \exp(ipt) \tag{9}$$

whence

$$y(t) = -i(C\alpha_0 U_0/mp) \exp(ipt) \tag{10}$$

Thus this first model of the rocking-wandering phenomenon predicts that the rocking and the wandering are sinusoidal motions at the rocking frequency of the sphere, and that the wandering lags the rocking by 90°. In each of our experiments both rocking and wandering were indeed almost sinusoidal motions at the sphere's rocking frequency. In some cases, when the frequency was not too large, the 90° phase shift was also observed (Fig. 3). However, especially at higher frequencies, the observed phase shift was greater than 90°. Also, as Fig. 3 illustrates, the rocking was always appreciably damped. Finally, it proved necessary to induce angular displacements on the order of one radian so that the rocking could be measured accurately, which casts doubt on the linearization of Eq. (6). To remove these discrepancies, we developed the modified theory presented in the following sections.

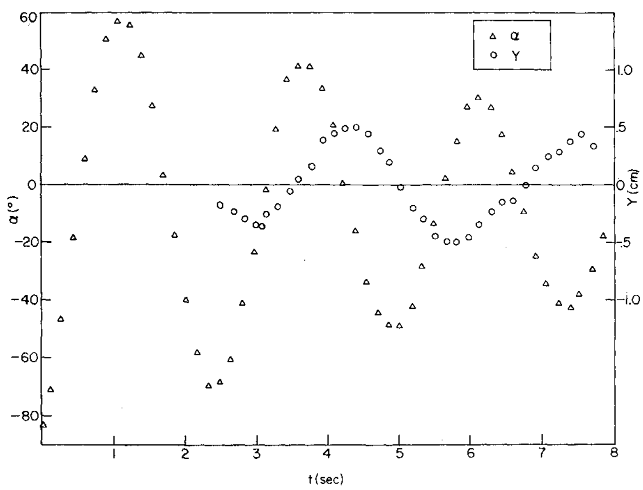


Fig. 3 Low-frequency rocking and wandering. Wandering lags rocking by 90°.

Modified Theory

The change in phase shift with frequency seems likely to arise because changes in ω do not in fact cause instantaneous changes in F, as required by Eq. (1). Presumably the change in F involves a rearrangement of the region of separated flow behind the sphere, or at least of the separation line on the sphere, which takes place over a significant time. To account for such a delay, we replaced Eq. (1) with the equations

$$F = CU_o q \tag{11}$$

$$\omega = q + \tau \dot{q} \tag{12}$$

These equations imply that if ω is changed suddenly from one constant value to another, F does not change abruptly, but tends to its new value with the factor $[1 - \exp(-t/\tau)]$. It must be emphasized that the modified relation (11, 12) between ω and F is an entirely artificial one, introduced to give some idea of the effect of a time lag between changes in ω and the corresponding changes in F.

Using Eqs. (8, 11, and 12), one sees that in the new model y and α are connected by the relation

$$d^3y/dt^3 + (1/\tau)d^2y/dt^2 = (v/\tau)d\alpha/dt \tag{13}$$

where the quantity

$$v \equiv CU_o/m \tag{14}$$

has the dimensions of a velocity.

The modified theory is completed by taking the pendulum equation with linear damping, instead of Eq. (6), to describe the rocking

$$I\ddot{\alpha} = -\delta mg \sin\alpha - \phi\dot{\alpha} \tag{15}$$

Boundary conditions appropriate to our experiments were

$$y(0) = 0, \quad \dot{y}(0) = 0, \quad F(0) = 0, \quad \alpha(0) = \alpha_o, \quad \dot{\alpha}(0) = 0 \tag{16}$$

The wandering determined from Eqs. (13, 15, and 16) is a damped oscillation about a straight line of negative slope. Such a negative drift was observed in all our experiments. The observed drifts were considerably smaller than those predicted by the theory, however, and the drifting eventually stopped, so that the oscillations took place about a vertical line shifted from the original free-fall line. Such a drift can be produced by the introduction of a drag term into the equation of the lateral motion, Eq. (8). We think, though, that the additional insight which might be gained by adding such a term to the present model does not justify the required effort. Also, the drift predicted by even a modified model will be strongly affected by the initial conditions. Whereas Eq. (16) is certainly a plausible set of initial conditions to use with a model, the actual beginning of the motion involves acceleration of the sphere, formation of a separated flow region behind the sphere, and transition of this region to turbulence. The transition is generally accompanied by a pronounced burst of vortex shedding.[14] Some effects of the starting process could be accounted for by modifying the present model. Again, however, we do not think such a change would materially improve one's understanding of the basic phenomenon, and we have removed the drift from both theoretical and experimental data in the following. We hoped that the resulting model would give acceptable predictions of the frequencies of the rocking and wandering, and of the phase shift between them, after an initial starting period, thus confirming that the basic mechanisms of the phenomenon had been identified. It was gratifying but puzzling to find that in many cases the model Eqs. (13, 15, and 16) also gave a fairly good point-by-point prediction of the observed motion, after a starting period of less than one period of the rocking. In the following sections the experiments and the comparisons between theory and experiments will be discussed in detail.

Description of the Experiments

To test the validity of the theoretical model of the wandering described above, biased spheres were released in a water filled tank and photographed with a motion picture camera. Data were obtained from frame-by-frame projection of the films, and included vertical and horizontal position and angular orientation.

The investigation was carried out in a plexiglass water tank 8 ft high and 1 ft square in cross section. The lower half of the tank was reinforced by aluminum angles at the corners, which were held in place by bands of steel wire separated by a distance of 2 in. The spheres were released at an angle of 90° from their statically stable orientation. The releasing mechanism consisted simply of a clamp constructed of spring steel, which held the sphere at its initial orientation until release.

Biased spheres were constructed by two different methods. The first was to fill table tennis balls with mixtures of solids of varying densities, ranging from wax through clay to metal. The most successful method of achieving low Reynolds number motion and low bias was to inject the sphere with a gelatin solution using a hypodermic needle. The solution would then solidify and thus produce the correct boundary condition at the inside of the sphere. The gelatin-filled spheres had specific gravities only slightly greater than one and hence produced relatively low Reynolds number motion. Liquid filled spheres could not be used since they do not satisfy the solid body condition of interest to us.

The second method was to construct a single sphere with variable bias. This was accomplished by first making a plexiglass sphere. A hole was then drilled on a diameter through the center of the sphere. This hole was filled with a round plexiglass rod and the sphere was reworked. In this way end caps were produced for the hollow core of the sphere. Small disks of the same diameter as the core hole were manufactured of plexiglass and steel. In this way it was possible to fill the core of the sphere with various combinations of plexiglass and steel disks, and produce various biases.

An attempt was made to visualize the flow about the rocking sphere by the use of water-soluble fluorescein dye. However, the dye method was not adequate for visualization of the time evolution of the entire flow. It was possible to show that at a Reynolds number of several thousand the wake behind a nonrocking freely falling sphere does not consist of a coherent recirculation region which is continually shed. On the contrary, after a shedding of vorticity during the initial acceleration, the wake appears to be laminar for almost its entire length and turbulent at its end. It is never entirely shed, although there is an exchange of fluid at the end of the wake. Fluid leaves the wake on one side and enters it on the other. This process is illustrated in Fig. 4, where it is evident that fluid is entering the wake on the right side and leaving it on the left.

Comparison between Theory and Experiment

To compare the model of the wandering sphere given by Eqs. (13, 15, and 16) with experiments, values of the damping factor ϕ and time constant τ are required. Since they are determined by the mean flow, τ and ϕ should be functions only of Reynolds number. One could obtain τ and ϕ by curve fitting to the observed motion at one rocking frequency for each Reynolds number, and check that these values led to satisfactory predictions for the motion at other frequencies for the same Reynolds number. However, this amounts to quite a bit of curve fitting. To provide a more demanding test of the model, involving less curve fitting, we determined τ and ϕ by curve fitting at only one Reynolds number (2.78×10^4) and one frequency ($8.544\ \text{sec}^{-1}$). We assumed that the damping would not vary markedly with Reynolds number, and the same damping factor was used to obtain theoretical

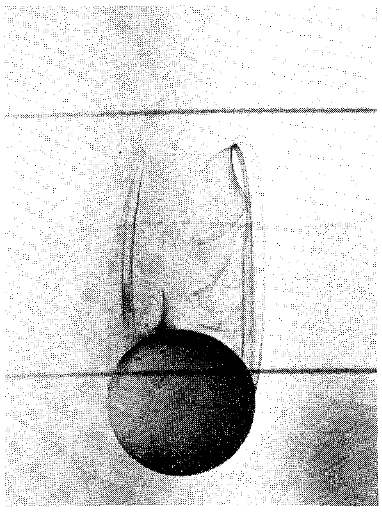


Fig. 4 Wake of freely falling sphere.

predictions for all runs. The time constant τ was scaled with Reynolds number by the following considerations:

We interpret τ as the time constant of some readjustment of the separated wake. We assume that

$$T = a/U_o \tag{17}$$

is the time scale of this readjustment, so that

$$\tau \propto T \tag{18}$$

Since

$$a/U_o = a^2/(\nu Re) \tag{19}$$

for spheres of equal radii falling in a given fluid,

$$\tau \propto T \propto 1/Re \tag{20}$$

The comparison between theory and experiment is shown in Figs. 5 and 6. In Fi6. 5, observed and predicted phase shifts are shown as functions of frequency, at different Reynolds numbers. It appears that the theory does predict qualitatively the connection between phase shift and frequency over the range of frequencies and Reynolds numbers tested, although agreement is poor at higher frequencies and Reynolds numbers. Figure 6 shows two examples of the comparison between predicted and observed trajectories and rocking motions for Reynolds numbers and frequencies considerably different from the one at which τ and ϕ were determined. The predicted rocking agreed well with observations in nearly every case, so that the assumed independence of ϕ from Reynolds number is better supported than is the assumed variation of τ as $1/Re$.

Occurrence of the Phenomenon

The coupling necessary for the wandering described in this paper does not occur if σ is too large. Essentially, if the rocking frequency is too high, the sphere rocks as it falls but does not wander. A rough estimate of the range of parameters over which bias-induced wandering occurs can be obtained from Maccoll's data, Fig. 1. From this picture it is clear that, for the steadily spinning sphere, lateral force is proportional to angular velocity only if

$$\sigma < \sigma_{mx}(Re) \tag{21}$$

While we have seen that there are differences between the periodic and steady-state cases, it still seems that the required coupling is unlikely to occur in the periodic case unless Eq. (21) is met, perhaps for a different $\sigma_{mx}(Re)$ than that of the steady-state case. Now, by assumption 3)

$$\rho_f U_o^2(2\pi a^2)C_D = 4\pi a^3(\rho_s - \rho_f)g/3$$

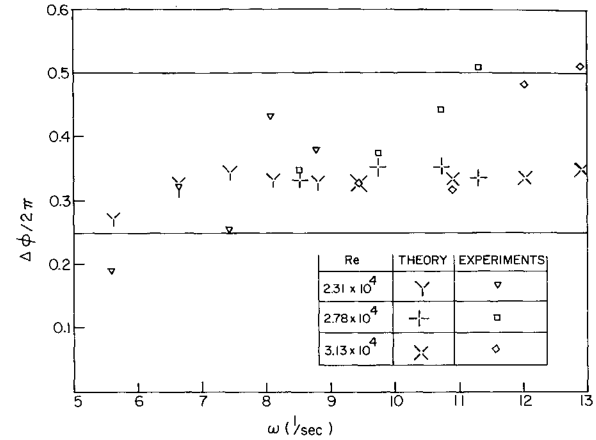


Fig. 5 Variation of the wandering's phase lag with frequency and Reynolds number. The upper and lower solid lines show the theoretical limiting values for large and small frequencies, respectively.

whence

$$U_o = [2a(\gamma - 1)g/(3C_D)]^{1/2} \tag{22}$$

Also, the amplitude of ω for the rocking sphere is

$$\omega^* = \alpha_o p = \alpha_o[5\delta g/(2a^2)]^{1/2} \tag{23}$$

Taking this value for ω to compute a σ for the rocking sphere, one has

$$\sigma = \alpha_o\{15C_D(Re)\beta/[4(\gamma - 1)]\}^{1/2} \tag{24}$$

Condition (21) then implies that wandering does not occur unless

$$15C_D(Re)\beta/[4(\gamma - 1)] < \sigma_{mx}^2(Re) \tag{25}$$

Alternately, if σ is too small, the lateral force may be so small that other perturbers, such as turbulence in the fluid, mask its effects. It is clear from Eq. (24), however, that if γ is very close to one, even a very small bias β will bring σ into the range where bias-induced wandering may be expected. On the other hand, inequality (25) indicates that the type of wandering considered here can be suppressed by making β sufficiently large, for given values of γ and C_D. This is

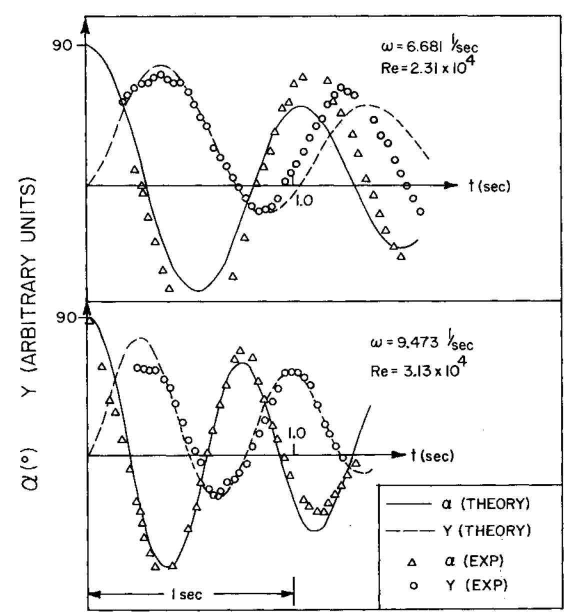


Fig. 6 Theoretical and experimental trajectories.

indeed so. In fact, Newton[1] made use of this effect to stabilize the trajectories of his weighted wax spheres.

The mechanism we have described will certainly be affected by transition at Reynolds numbers around 10^5, and by the possibility of a stably oscillating recirculation region at Reynolds numbers below a few hundred.[15] Therefore the phenomenon considered here is to be expected mostly at moderate Reynolds numbers ($10^3 < Re < 10^5$) and low mass ratios ($0.8 \leq \gamma \leq 1.2$).

We remark in passing that we observed no evidence of the negative lift observed—and questioned—by Maccoll for $\sigma < 0.5$, even though some of our experiments were at $\sigma = 0.16$.

Summary and Conclusions

We find sufficiently good qualitative agreement between observed motions of wandering, falling spheres and the present theoretical model, in which the wandering is ascribed to coupling between bias-induced rocking of the spheres and their lateral motion, to conclude that this model does reveal the basic mechanisms of the wandering at certain combinations of Reynolds number, frequency, and sphere-to-fluid density ratio. The bias-induced wandering described above is more likely to occur for small, inadvertent bias if the sphere is only slightly more (less) dense than the fluid in which it falls (rises). Inequality (25) gives some indication of whether or not bias-induced wandering will occur. The assumptions that damping of the sphere's rocking is independent of Reynolds number, while the time constant for the response of the lateral force on the sphere to changes in its angular velocity is inversely proportional to Reynolds number, lead to fair agreement with experiments over the range $2.18 \times 10^4 < Re < 3.13 \times 10^4$, which seems noteworthy in view of the complexity of the flow.

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